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A WESTERGAARD-TYPE STRESS FUNCTION FOR LINE INCLUSION PROBLEMS

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Abstract—A direct correspondence is shown to exist between the Westergaard stress function for crack problems and a newly-introduced stress function for rigid line inclusion problems. A correspondence is also shown to exist between stress intensities in the crack and the inclusion problems, as well as between the opening displacement of the crack and the axial force in the inclusion. A scheme is presented to modify these rigid-inclusion solutions to account approximately for non-zero compliance of real fibers in a composite material.

1. INTRODUCTION

An understanding of the mechanics of force transfer between the fibers and the matrix is an essential step toward failure prediction in fiber-reinforced composite materials. The mechanics of force transfer between fibers of finite length and a surrounding matrix have been investigated extensively by both experimental[1–4] and theoretical means[5–19].

One of the popular approaches to the study of this problem has been to represent the long thin fibers by line inclusions[8–19]. With this idealization the fiber cross-sectional area vanishes so that the mathematical fiber does not occupy space in the matrix, but simply introduces a line discontinuity in the matrix stress field. At the same time the idealized fiber retains an axial stiffness.

The line inclusion model has been used to study stress fields associated with cracking in the fiber matrix interface[8–10], interaction of two or more adjacent fibers[9–16], and various fiber end shapes[17]. In addition, problems involving a broken fiber[10], a fiber parallel to a bi-material interface[18], a fiber directed normal to a free surface[16], and a crack approaching a fiber[20] have been considered. Even without cracking the stress field exhibits a square root singularity in the neighborhood of the tip of the line inclusion. For this reason, Erdogan[21] and others[10, 11] have suggested the use of the inclusion-tip stress intensity factor as a parameter for predicting crack initiation in composite materials.

Sih[11] and Atkinson[16] present solutions for which the line inclusions are rigid. These results are relevant to composites in view of the high ratio of the fiber and matrix moduli common in real fiber reinforcements. Furthermore, the solutions presented in [11, 16] are particularly useful for studying fiber-matrix force transfer, because the stress fields have concise closed form expressions.

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As Sih[11] indicates, geometric similarity between crack and line inclusion problems permits similar solution methods to be employed. A useful method to obtain solutions to certain two-dimensional crack problems involves the use of the Westergaard stress function.† Solutions based on this stress function are available in the literature for a large number of relevant crack geometries (see, e.g. Tada[24]).

In [16] Atkinson declares that the solution to a problem of two equal, collinear, rigid line inclusions can be written down by observing an analogy with the known solution for the geometrically similar crack problem. Beyond this single application however, Atkinson offers no further explanation of this analogy.

In the following, the analogy between problems of rigid line inclusions and traction free cracks, alluded to by both Sih and Atkinson, is clarified. A direct correspondence is shown to exist between the Westergaard stress function for crack problems and a newly introduced stress function for line inclusion problems. A correspondence is also demonstrated between stress intensity factors at the crack and inclusion tips. In addition, a correspondence between the crack opening displacement and the fiber axial force is presented.

Section 2 is devoted to certain mathematical preliminaries and a brief review of the Westergaard stress function. In Section 3 a stress function for line inclusions is introduced and the above mentioned relations with the Westergaard stress function are presented. Section 4 is concerned with fibers with non-zero axial compliance and a method to approximately account for this effect. An example is presented and compared with an earlier result obtained numerically. Section 5 contains the concluding remarks.

2. MATHEMATICAL PRELIMINARIES

As discussed in depth by Muskhelishvili[25], the stress and displacement fields for two-dimensional problems in classical elastostatics may be expressed in terms of Kolosov–Goursat functions Φ and Ω. Let $\sigma_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ be the relevant components of the stress tensor while u_α denotes the components of the displacement vector. Suppose a linearly elastic body with shear modulus μ and Poisson's ratio ν occupies a region S. Further, let S be a region of analyticity of Φ and Ω. Then, when symmetry exists about the axis $x_2 = 0$ [25],

$$\sigma_{11} + \sigma_{22} = 4\mathrm{Re}\,[\Phi(z)] \tag{1a}$$

$$\sigma_{22} - i\sigma_{12} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\Phi'(\bar{z}) \tag{1b}$$

$$2\mu(u_1 + iu_2)_{,1} = \kappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\Phi'(\bar{z}) \tag{1c}$$

where the comma denotes partial differentiation with respect to the following subscript, prime represents differentiation with respect to the complex variable $z = x_1 + ix_2$, and

$$\kappa = \begin{cases} (3 - 4\nu) & \text{plane strain} \\ (3 - \nu)/(1 + \nu) & \text{generalized plane stress} \end{cases} \tag{2}$$

In the following it is assumed that the stress and strain states are bounded as $|z| \to \infty$. The limiting values of the normal components are related by Hooke's law

$$8\mu \begin{Bmatrix} u_{1,1}(\infty) \\ u_{2,2}(\infty) \end{Bmatrix} = \begin{bmatrix} (\kappa + 1) & (\kappa - 3) \\ (\kappa - 3) & (\kappa + 1) \end{bmatrix} \begin{Bmatrix} \sigma_{11}(\infty) \\ \sigma_{22}(\infty) \end{Bmatrix} \tag{3}$$

†This stress function was originally introduced by MacGregor[22] while Westergaard[23] first applied it to the solution of crack problems.

where

$$\sigma_{\alpha\beta}(\infty) \equiv \lim_{|z|\to\infty} \sigma_{\alpha\beta}.$$

Symmetry about the real axis $x_2 = 0$ requires that σ_{12}, $u_{1,2}$ and $u_{2,1}$ vanish as $|z| \to \infty$.

Among others, Muskhelishvili[25] considers the class of problems in which n segments of discontinuity lie along the real axis. Denote by L the set of these discontinuities and by S the entire z plane cut along L. As shown in Fig. 1, the jth segment of L extends from $x_1 = b_j - a_j$ to $x_1 = b_j + a_j$, $j = 1, 2, \ldots, n$, along the real axis.

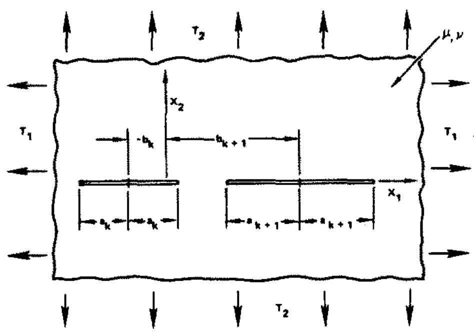

Fig. 1. General geometry.

Now let L correspond to a set of cracks with zero tractions on the crack surfaces; i.e.

$$\sigma^+_{\alpha 2}(x_1) = \sigma^-_{\alpha 2}(x_1) = 0 \qquad x_1 \epsilon L, \qquad \alpha = 1, 2 \tag{4}$$

where superscripts (+) and (−) denote the limiting values as a point on L is approached from the directions $x_2 > 0$ and $x_2 < 0$ respectively. From equations (120.16) and (120.18) of [25] the solution in S is given by

$$\Phi(z) - \Omega(z) = A \tag{5a}$$

$$\Phi(z) + \Omega(z) = Z(z) - A \tag{5b}$$

where A is the real constant

$$A = \frac{1}{2}[\sigma_{11}(\infty) - \sigma_{22}(\infty)]. \tag{6}$$

The function $Z(z)$ in (5b) is the Westergaard stress function given by

$$Z(z) = \frac{\sigma_{22}(\infty)}{X(z)}\left(z^n + \sum_{j=1}^{n} C_j z^{j-1}\right) + A \tag{7}$$

where

$$X(z) = \prod_{j=1}^{n} (z - b_j - a_j)^{1/2}(z - b_j + a_j)^{1/2} \tag{8}$$

and C_j are real-valued coefficients. From equations (1) and (5) the stresses and displacements in terms of the Westergaard stress function are determined to be

$$\sigma_{11}+\sigma_{22}=2Re[Z(z)] \tag{9a}$$

$$\sigma_{22}-i\sigma_{12}=Re[Z(z)]+\frac{(z-\bar{z})}{2}Z'(\bar{z})-A \tag{9b}$$

$$2\mu(u_1+iu_2)_{,1}=\frac{1}{2}\{\kappa Z(z)-Z(\bar{z})-(z-\bar{z})Z'(\bar{z})\}+A. \tag{9c}$$

Let $\Delta_k(x_1)$ denote the opening displacement of the crack occupying the kth segment of L. Then from (9c)

$$\Delta_k(x_1)=2u_2^+(x_1)=\frac{(\kappa+1)}{2\mu}Im\int_{b_k-a_k}^{x_1}Z^+(t)\,dt \quad x_1-b_k\epsilon(-a_k,a_k);\quad k=1,2,\ldots,n. \tag{10}$$

To complete the solution, the real valued coefficients C_j in equation (7) are evaluated by enforcing the n side conditions requiring the opening displacement of every crack to be zero at the crack tips, $x_1=b_k+a_k$. Substitution from (7) into (10) provides

$$Im\int_{b_k-a_k}^{b_k+a_k}\frac{1}{X^+(t)}\left(t^n+\sum_{j=1}^{n}C_jt^{j-1}\right)dt=0 \quad k=1,2,\ldots,n. \tag{11}$$

3. A STRESS FUNCTION FOR RIGID LINE INCLUSION PROBLEMS

An identical approach to that in Section 2 can be followed if L is occupied by rigid line inclusions instead of traction free cracks. In this class of problems a perfect bond is assumed to exist along the inclusion interface, so that the displacements on L vanish. Consequently

$$u^{\pm}_{\alpha,1}(x_1)=0 \quad x_1\epsilon L \quad \alpha=1,2. \tag{12}$$

Using equations (120.34) and (120.35) of [25] leads to the following solution for Φ and Ω.

$$\kappa\Phi(z)+\Omega(z)=A_1 \tag{13a}$$

$$\kappa\Phi(z)-\Omega(z)=Z_1(z)-A_1. \tag{13b}$$

In (13a,b), A_1 is a real valued constant given by

$$A_1=\frac{1}{4}[(\kappa-1)\sigma_{11}(\infty)+(\kappa+3)\sigma_{22}(\infty)] \tag{14}$$

while the new stress function, $Z_1(z)$, is

$$Z_1(z)=\frac{2\mu u_{1,1}(\infty)}{X(z)}\left(z^n+\sum_{j=1}^{n}D_jz^{j-1}\right)+A_1 \tag{15}$$

where $X(z)$ has been defined in equation (8). The constants D_j are real values and remain to be determined. The stress and displacement field is readily evaluated from equations (1) and (13) to give

$$\sigma_{11}+\sigma_{22}=\frac{2}{\kappa}Re[Z_1(z)] \tag{16a}$$

$$\sigma_{22}-i\sigma_{12}=\frac{1}{2\kappa}\{Z_1(z)-\kappa Z_1(\bar{z})+(z-\bar{z})Z_1'(\bar{z})\}+A_1 \tag{16b}$$

$$2\mu(u_1+iu_2)_{,1}=Re[Z_1(z)]-(z-\bar{z})\frac{1}{2\kappa}Z_1'(\bar{z})-A_1. \tag{16c}$$

The axial force in the embedded fiber is of physical interest in composites. These axial forces result from the transfer of shear along the matrix fiber interfaces. Denoting the axial force in the jth fiber by $P_j(x_1)$ and using (16b) leads to the following expression

$$P_j(x_1)=\left(\frac{\kappa+1}{\kappa}\right)s_j Im\int_{b_j-a_j}^{x_1}Z_1^+(t)\,dt+P_j(b_j-a_j)\quad x_1-b_j\epsilon(-a_j,a_j) \tag{17}$$

where s_j is the specific area (area per unit length) of the lateral interface between the matrix and the jth inclusion.

The constant $P_j(b_j-a_j)$ is the force transferred to the fiber at its end $x_1=b_j-a_j$. Muki and Sternberg[26] have proved that the axial force at the embedded end of a line inclusion must vanish, verifying an earlier conjecture by Reissner[27]. Thus $P_j(b_j-a_j)=0$ in (17).

This requirement that the fibers have zero axial force at their end points permits the evaluation of the constants D_j in equation (15). Thus from (15) and (17) the following conditions result

$$Im\int_{a_k-b_k}^{a_k+b_k}\frac{1}{X^+(t)}\left(t^n+\sum_{j=1}^{n}D_jt^{j-1}\right)dt=0\quad k=1,2,\ldots,n \tag{18}$$

permitting the complete determination of the stress function $Z_1(z)$.

It is now straightforward to demonstrate the correspondence between the Westergaard stress function $Z(z)$ for crack problems and the stress function $Z_1(z)$ introduced here for line inclusion problems.

To begin, select a particular geometry; that is let n, b_j, and a_j in Fig. 1 be fixed. Then comparison of equations (10) and (18) shows that

$$C_j=D_j\quad j=1,2,\ldots,n. \tag{19}$$

In light of (19), equations (7) and (15) indicate that the stress functions for the two classes of problems are related by

$$\frac{Z(z)-A}{\sigma_{22}(\infty)}=\frac{Z_1(z)-A_1}{2\mu u_{1,1}(\infty)}\equiv Z_0(z) \tag{20}$$

where

$$Z_0(z)=\frac{1}{X(z)}\left(z^n+\sum_{j=1}^{n}C_jz^{j-1}\right). \tag{21}$$

In the same way a correspondence between the crack opening displacement Δ_j and the inclusion axial force P_j can be shown. Comparing equations (10) and (17) and using (20) leads to

$$\frac{2\mu\Delta_j(x_1)}{(\kappa+1)\sigma_{22}(\infty)}=\frac{\kappa P_j(x_1)}{(\kappa+1)s_j 2\mu u_{1,1}(\infty)}=\Psi_j(x_1) \tag{22}$$

where

$$\Psi_j(x_1)=Im\int_{b_j-a_j}^{x_1} Z_0^+(t)\,dt \quad x_1-b_j\epsilon(-a_j,a_j) \quad j=1,2,\ldots,n. \tag{23}$$

Finally an equivalence between the crack-tip and inclusion-tip stress intensity factors can be developed. Let r denote a small distance from the tip of the crack or line inclusion into S. It can then be shown for both problems that the stress field is proportional to $r^{-1/2}$ as $r\to 0+$. The factor of proportionality is known as the stress intensity factor and for cracks is given by

$$K_I\equiv\lim_{r\to 0}[\sqrt{(2\pi r)}Z^+(x_1)] \quad x_1\notin L. \tag{24}$$

In a similar manner, a stress intensity factor $K_I^{(1)}$ can be defined for line inclusion problems;

$$K_I^{(1)}\equiv\lim_{r\to 0}[\sqrt{(2\pi r)}Z_1^+(x_1)] \quad x_1\notin L. \tag{25}$$

Accordingly the stress field as $r\to 0$ at an angular orientation θ from the inclusion tip, where $|\theta|\epsilon[0,\pi]$ and θ is the clockwise orientation (in Fig. 1) away from the ray $z=x_1, x_1\notin L$ is given by†

$$\sigma_{12}=\frac{K_I^{(1)}}{2\kappa\sqrt{(2\pi r)}}[(\kappa+1)\sin(\theta/2)+\sin(\theta)\cos(3\theta/2)]+0(1) \quad \text{as } r\to 0+ \tag{26a}$$

$$\sigma_{11}=\frac{K_I^{(1)}}{2\kappa\sqrt{(2\pi r)}}[(3+\kappa)\cos(\theta/2)-\sin(\theta)\sin(3\theta/2)]+0(1) \quad \text{as } r\to 0+ \tag{26b}$$

$$\sigma_{22}=\frac{K_I^{(1)}}{2\kappa\sqrt{(2\pi r)}}[-(\kappa-1)\cos(\theta/2)+\sin(\theta)\sin(3\theta/2)]+0(1) \quad \text{as } r\to 0+. \tag{26c}$$

In view of (20), comparison of equations (24) and (25) shows that the stress intensity factors in the two classes of problems are related by

$$\frac{K_I}{\sigma_{22}(\infty)}=\frac{K_I^{(1)}}{2\mu u_{1,1}(\infty)}\equiv\alpha=\lim_{r\to o}[\sqrt{(2\pi r)}Z_0^+(x_1)] \quad x_1\notin L. \tag{27}$$

As one example of the above analogy, consider the configuration shown in Fig. 2. The problem consists of an infinite number of equal length, equally spaced, rigid, line inclusions located on the real axis. Here $n\to\infty$, while $b_j=2jb$ and $a_j=a$, $j=0,1,2,\ldots$. From the well known crack problem solution by Westergaard it is determined that

$$Z_0(z)=\left\{1-\frac{\sin^2(\pi a/2b)}{\sin^2(\pi z/2b)}\right\}^{-1/2} \tag{28}$$

†Equations (26) are equivalent to expressions given by Sih[11], where Sih's k_1 is equal to $K_I^{(1)}/\sqrt{\pi}$.

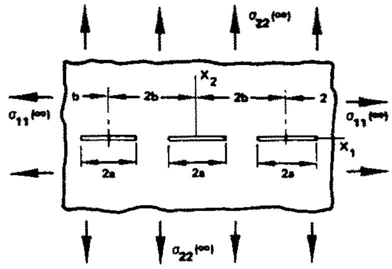

Fig. 2. Infinite row of equal, equally-spaced cuts.

and

$$\Psi(x_1) = \frac{2b}{\pi} \ln \left\{ \frac{\cos(\pi x_1/2b)}{\cos(\pi a/2b)} + \sqrt{\left(\frac{\cos^2(\pi x_1/2b)}{\cos^2(\pi a/2b)} - 1 \right)} \right\} \tag{29}$$

$$\alpha = \sqrt{(2b \tan(\pi a/2b))}. \tag{30}$$

Using equations (20), (22) and (27) in conjunction with (28), (29) and (30) permits the immediate calculation of the stress function $Z_1(z)$, axial force $P_f(x_1)$, and stress intensity factor $K_I^{(1)}$. This solution agrees with Sih's[11] solution obtained in terms of Kolosov–Goursat functions. Other examples illustrating the application of the analogy are given in [10].

In [16] Atkinson follows a somewhat similar approach to the development herein. He defines a new function $w'(z)$ to replace $\Omega(z)$ in equations (1) such that for line inclusion problems, $w'(z)$ is real on the entire x_1 axis, analytic on $S \cup L$, and consequently, representable by a polynomial. In his example problems involving line inclusions in the unbounded domain, Atkinson sets $w'(z) = 0$. However, the far-field conditions are not satisfied unless $w'(z) = A_1$. This minor oversight by Atkinson is similar to MacGregor's[22] and subsequently Westergaard's[23] omission† of the constant A in their versions of equation (9). The omission of this constant was corrected in a paper by Sih[28] and discussed further by Eftis and Liebowitz[29].

4. EFFECT OF NON-ZERO FIBER COMPLIANCE

Using the above analogy, it is possible to obtain the solution for a number of problems involving perfectly rigid line inclusions. Any real composite, however, will contain fibers with an elastic modulus which, although large compared to that of the matrix, is finite. If the relatively simple solutions for rigid fibers are to be used in practical applications, a method to account for finite fiber stiffness is required.

A basis for an approximate approach is provided by a problem considered by Atkinson[16] and others[17–19]. The problem, Fig. 3, involves a single elastic line inclusion of length $2a$ in a two-dimensional matrix subjected to a biaxial field $\sigma_{11}(\infty)$, $\sigma_{22}(\infty)$. The line inclusion represents an elastic fiber with Young's modulus E_f, cross-sectional area A_f, and fiber-matrix interface surface area $2as_1$. It is found convenient to characterize the compliance of the fiber by a dimensionless parameter γ_E where

$$\gamma_E = \frac{(\kappa + 1)as_1\mu}{\pi\kappa E_f A_f}. \tag{31}$$

†Westergaard's omission of A is not really an error since he correctly restricts his remarks to pressurized cracks for which the far field stress state vanishes so that $A = 0$.

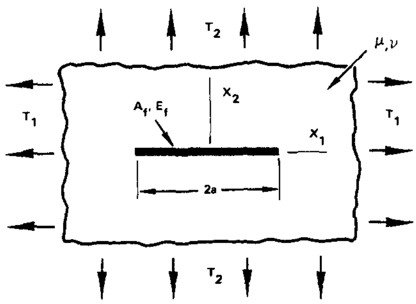

Fig. 3. Elastic line inclusion.

Treating the interface shear stress $\overset{+}{\sigma}_{12}(x_1)$, $|x_1| < a$, as the unknown, Atkinson[16] obtains an asymptotic solution for $\gamma_E \ll 1$. He also formulates the problem in terms of a Fredholm integral equation of the second kind. This approach is applicable for all $\gamma_E \geq 0$, but Atkinson does not solve the equation. As discussed in Appendix A of [10], Brussat has independently obtained the same integral equation formulation and computed a numerical solution.

The solution from [10] is summarized in Fig. 4, where the effect of fiber compliance is reflected in the function $\lambda(x_1/a, \gamma_E)$ defined by

$$\lambda(x_1/a, \gamma_E) = \frac{\overset{+}{\sigma}_{12}(x_1; \gamma_E)}{\overset{+}{\sigma}_{12}(x_1; 0)} \quad |x_1| < a \tag{32a}$$

$$\lambda(1, \gamma_E) = K_I^{(1)}(\gamma_E)/K_I^{(1)}(0). \tag{32b}$$

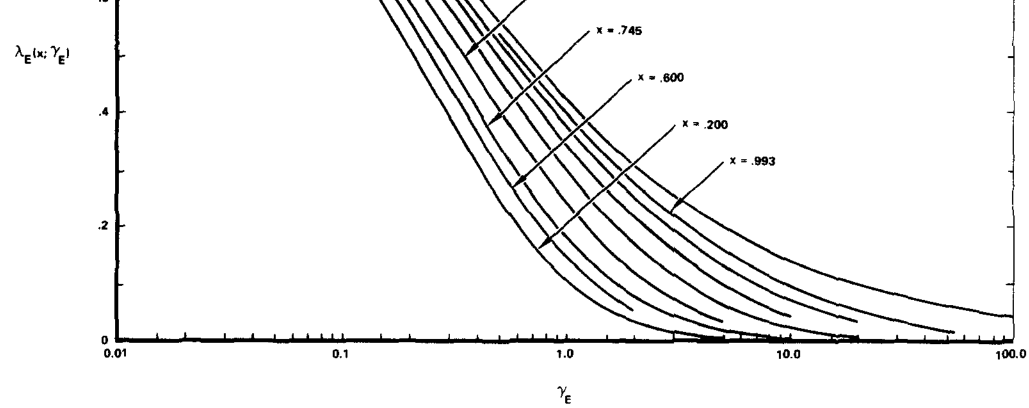

Fig. 4. Fiber compliance function.

Here $\sigma^+_{12}(x_1; \gamma_E)$ and $K_I^{(1)}(\gamma_E)$ are the interface shear stress and stress intensity factor for the elastic line inclusion whereas $\sigma^+_{12}(x_1; 0)$ and $K_I^{(1)}(0)$ are the corresponding quantities for the single, rigid line inclusion given by

$$\sigma^+_{12}(x_1; 0) = \frac{(\kappa + 1)}{\kappa}\mu u_{1,1}(\infty)\frac{x_1}{\sqrt{(a^2 - x_1^2)}} \tag{33}$$

$$K_I^{(1)}(0) = 2\mu u_{1,1}(\infty)\sqrt{(\pi a)}. \tag{34}$$

To exactly account for the non-zero fiber compliance in the more general case requires the numerical solution of a system of Fredholm integral equations. As this is somewhat time consuming, an approximate procedure is now suggested to estimate this effect.

Consider the problem of the cut plane with an arbitrary set L of cut segments on the real axis containing *elastic* line inclusions. Let $\sigma^+_{12}(x_1; 0)$ and $K_I^{(1)}(0)$ denote the interface shear stress and stress intensity factors for the same geometry except that the inclusions are rigid. This rigid inclusion problem is readily solved by the method indicated in Section 3.

The approximate solution for the interface shear stresses and intensity factors in the elastic line inclusion problem is now obtained from the following:

$$\sigma^+_{12}(x_1; \gamma_E) \doteq \lambda(x_1/a, \gamma_E)\sigma^+_{12}(x_1; 0) \tag{35}$$

$$K_I^+(\gamma_E) \doteq \lambda(1, \gamma_E)K_I^{(1)}(0) \tag{36}$$

where λ is given in Fig. 4. Provided the fibers are widely spaced, it is expected that equations (35) and (36) will yield accurate answers. As the fiber spacing decreases, the adjacent fibers will interact and the results deviate from the approximate prediction given by equations (35) and (36).

To assess the accuracy of this approach an example is presented. In the absence of any exact 2-D solutions involving interacting flexible line inclusions, a numerical 3-D solution obtained by Carrara and McGarry [5] is considered. In [5] they used the finite element method to evaluate the interface shear stresses for the axisymmetric fibermatrix problem shown in Fig. 5. Carrara and McGarry let the inner cylinder, representing the fiber, have unit diameter. The innercylinder length was taken to be 28, the elastic modulus 11×10^3 ksi, and Poisson's ratio 0·22. The outer cylinder, representing the matrix, had diameter 10, length 38, Young's modulus $0{\cdot}6 \times 10^3$ ksi, and Poisson's ratio 0·34. Very low modulus finite elements were placed in the matrix just beyond the ends of the fiber to prevent load transfer across the fiber ends. Uniform axial strain was enforced at the ends of the larger cylinder.

The two-dimensional approximation to this configuration is the infinite row of equally-spaced,

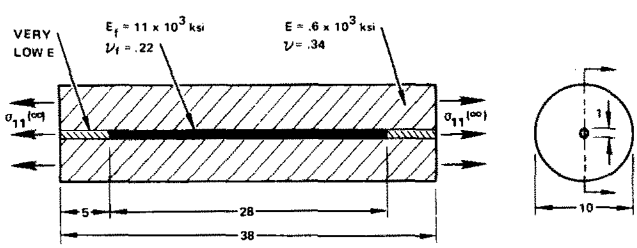

Fig. 5. Axisymmetric problem [5].

equal-length, collinear line inclusions shown in Fig. 2. The conditions along lines of symmetry between adjacent fibers in this problem are

$$\sigma_{12}(z_k) = 0 \tag{37a}$$

$$u_1(z_k) = u_{1,1}(\infty)(2k+1)b \tag{37b}$$

$$\text{where } z_k = b(2k+1) + ix_2;\ k = 0, \pm 1, \pm 2, \ldots;\ x_2 \epsilon (-\infty, \infty). \tag{38}$$

Therefore, this problem is equivalent to the problem of a strip, infinite in the x_2-direction and of width $2b = 38$ in the x_1 direction. The strip contains an elastic fiber of length $2a = 28$ stretched uniformly between parallel frictionless planes which rigidly resist bending.

The rigid fiber solution for this case is given in Section 3. From (16) and (28) the interface shear stress for $\gamma_E = 0$ is determined to be

$$\sigma^{+}_{12}(x_1; 0) = \frac{(\kappa+1)}{\kappa}\mu u_{1,1}(\infty)\left\{1 - \frac{\sin^2(\pi x_1/2b)}{\sin^2(\pi a/2b)}\right\}^{-1/2}. \tag{39}$$

In the case of plane strain and $\nu = 0{\cdot}34$, $\kappa = 1{\cdot}64$. Using equation (30) with $s_1 = \pi$ and $A_f = \pi/4$ leads to $\gamma_E = 0{\cdot}584$. The function $\lambda(x_1/a, 0{\cdot}584)$ can now be obtained from Fig. 4 and the approximate interface shear stress calculated from (35). This estimate is plotted in Fig. 6 and compared with the finite element result for the three dimensional axisymmetric case[5].

The approximate two-dimensional solution is in excellent agreement with the three-dimensional results except within a distance from the end of about one-third the fiber diameter. A difference between the two solutions is expected in the end region as the fiber end shape influences the detailed stress distribution in this area. Also, the finite element approximation leads to a bounded stress field at the fiber tip while the line inclusion stress field is singular.

Although the agreement observed in Fig. 6 is encouraging, it would be imprudent to draw broad conclusions from a single success. A more fundamental check of the approximation is

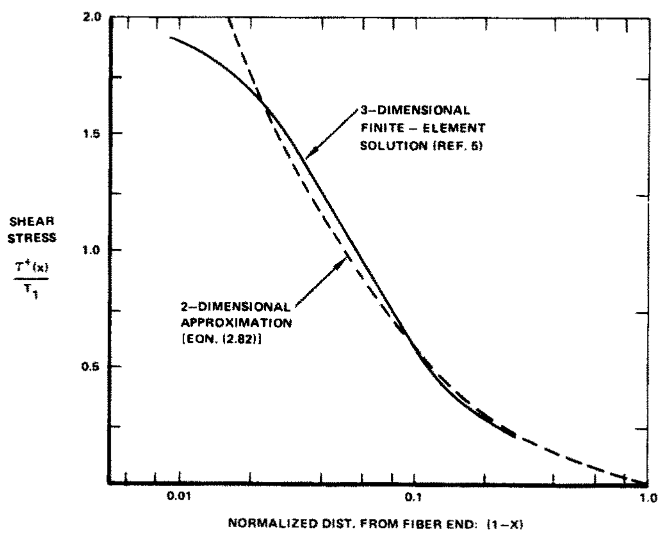

Fig. 6. Interface shear stress.

required. It is suggested that accurate numerical solutions for elastic fiber inclusions be obtained and the results compared with the approximate method presented here.

It is worth noting that a comparison similar to the one in Fig. 6 is made in [17] between a two-dimensional solution and the axisymmetric result of [5]. However, the two-dimensional approximation in [17] incorporates assumptions different from those adopted here. In [17], Bürgel, Perry and Schneider assumed: (a) Plane stress instead of a state of plane strain. (b) Fiber cross-section is a 1×1 rectangle, and the fiber contacts the matrix on only two of its four long sides. As a result $s_1/A_f = 2$ instead of $s_1/A_f = 4$ (the result for a circular cross-section in contact all around the circumference). (c) Fiber is isolated from all boundaries ($a/b = 0$ instead of $a/b = 14/19$).

Interestingly, these three assumptions have large but somewhat cancelling effects, so that the agreement obtained in [17] is almost as good as that shown in Fig. 6.

5. CONCLUDING REMARKS

An analogy has been shown to exist between the well known Westergaard stress function for line crack problems and a similar stress function for rigid line inclusion problems. As a result, closed form solutions to various two-dimensional elasticity problems involving rigid line inclusions can be generated from the Westergaard stress function for the similar traction-free line crack problem.

Certain extensions of the analogy are possible. For example, the problem of two semi-infinite cracks on the real axis, the ends of each a distance a from the origin, subjected to a remote force $F_2(\infty)$ normal to the cracks, i.e.

$$F_2(\infty) \equiv \int_{-\infty+i\gamma}^{\infty+i\gamma} \sigma_{22}(z)\,\mathrm{d}z \tag{40}$$

(where γ is any real constant) has the solution[24]

$$Z(z) = \frac{F_2(\infty)}{\pi} \frac{1}{\sqrt{(a^2 - z^2)}}. \tag{41}$$

The corresponding problem of two semi-infinite rigid line inclusions on the real axis, the tips of each a distance a from the origin, subjected to a total relative displacement $u_1(\infty)$ parallel to the inclusions; i.e.

$$2\mu u_1(\infty) = \int_{-\infty+i\gamma}^{\infty+i\gamma} 2\mu u_{1,1}(z)\,\mathrm{d}z \tag{42}$$

has the solution

$$Z_1(z) = \frac{2\mu u_1(\infty)}{\pi} \frac{1}{\sqrt{(a^2 - z^2)}}. \tag{43}$$

Thus, for this pair of problems, from (41) and (43),

$$\frac{Z(z)}{F_2(\infty)} = \frac{Z_1(z)}{2\mu u_1(\infty)}. \tag{44}$$

In view of (40) and (42) and since $A = A_1 = 0$, equation (44) strongly resembles (20). Similar extensions of the analogy to crack problems may permit direct solution of other rigid line inclusion problems.

As discussed above, the applicability of these solutions to fiber composites can be improved using the solution given in Fig. 4, together with equations (35) and (36), to approximately account for the effect of non-zero fiber compliance. This approximate two-dimensional procedure has been applied to the configuration shown in Fig. 5 and yields good agreement with the three-dimensional finite element analysis (Fig. 6).

The stress intensity factor $K_I^{(1)}$ for the fiber tip computed in this manner is expected to have the same relevance to crack initiation at the fiber tip as the fracture mechanics stress intensity factor K_I has to crack initiation at a crack tip.

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REFERENCES

1. W. R. Tyson and G. J. Davies, A photoelastic study of the shear stresses associated with the transfer of stress during fiber reinforcement. *Br. J. Appl. Phys.* **16**, 199 (1965).
2. T. F. MacLaughlin, Effect of fiber geometry on stress in fiber-reinforced composite materials. *Exper. Mech.* **6**, 481 (1966).
3. D. M. Schuster and E. Scala, The mechanical interaction of sapphire whiskers with a birefringent matrix. *Trans. Met. Soc. AIME* **230**, 1635 (1964).
4. I. M. Allison and L. C. Hollaway, The stresses around fibers in brittle matrices, *Plastics and Polymers* **37**, 57 (1969).
5. A. S. Carrara and F. J. McGarry, Matrix and interface stresses in a discontinuous fiber composite model. *J. Composite Mat'ls.* **2**, 222 (1968).
6. G. J. Schneider and H. D. Conway, Effect on fiber geometry and partial debonding on fiber-matrix bond stresses. *J. Composite Mat'ls.* **3**, 116 (1969).
7. H. L. Cox, The elasticity and strength of paper and other fibrous materials. *Br. J. Appl. Phys.* **3**, 72 (1952).
8. G. P. Sendeckyj, *Debonding of Rigid Inclusions in Plane Elastostatics*, AFFDL-TM-73-24-FBC. Air Force Flight Dynamics Laboratory, Wright–Patterson AFB, Ohio (March 1973).
9. H. D. Conway, W. W. Chu and C. I. Chang, Effect of bond stresses of partial bond failure of overlapping fibres in a composite material. *Fibre Sci. Tech.* **2**, 289 (1970).
10. T. R. Brussat, Line inclusion models of failure mechanisms in fiber composites. Ph.D. Dissertation, University of California, Los Angeles (1973).
11. G. C. Sih, Plane extension of rigidly embedded line inclusions. *Developments in Mechanics* (Edited by T. C. Huang and M. W. Johnson), Vol. 3, pp. 61–79. Wiley, New York (1965).
12. L. J. Cohen and J. P. Romualdi, Stress, strain and displacement fields in a composite material reinforced with discontinuous fibers. *J. Franklin Inst.* **284**, 388 (1967).
13. C. S. Chang and H. D. Conway, Bond stresses in fiber reinforced composites subjected to uniform tension, *J. Composite Mat'ls.* **2**, 168 (1968).
14. W. W. L. Chu and H. D. Conway, Bond stresses in composites with overlapping fibers. *Int. J. Mech. Sci.* **12**, 761 (1970).
15. H. D. Conway and C. I. Chang, The effective elastic constants and bond stresses for a fiber-reinforced elastic sheet. *Fibre Sci. Tech.* **3**, 249 (1971).
16. C. Atkinson, Some ribbon-like inclusion problems. *Int. J. Engrg. Sci.* **11**, 243 (1973).
17. B. Bürgel, A. J. Perry and W. R. Schneider, On the theory of fiber strengthening. *J. Mech. Phys. Solids* **18**, 101 (1970).
18. F. Erdogan and G. D. Gupta. Stresses near a flat inclusion in bonded dissimilar materials. *Int. J. Solids Struct.* **8**, 533 (1972).
19. H. Bufler, Scheibe Mit Endlicher, Elastischer Versteifung. VDI-Forschungscheft 485, BDI Verlag GMBH/Dusseldorf (1961).
20. R. Greif and J. L. Sanders, Jr., The effect of a stringer on the stress in a cracked sheet. *ASME Trans: J. Appl. Mech.* **32**, 59 (1965).
21. F. Erdogan, Fracture problems in composite materials. *Engrg. Fracture Mech.* **4**, 811 (1972).
22. C. W. MacGregor, The potential function method for the solution of two-dimensional stress problems. *Trans. Am. Math. Soc.* **38**, 177 (1935).
23. H. M. Westergaard, Bearing pressures and cracks. *ASME Trans: J. Appl. Mech.* **61**, A49 (1939).
24. H. Tada, *The Stress Analysis of Cracks Handbook*. Del Research Corp., Hellertown, Pennsylvania (1973).
25. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoff, Holland (1963).

26. R. Muki and E. Sternberg, On the diffusion of load from a transverse tension bar into a semi-infinite elastic sheet. *ASME Trans: J. Appl. Mech.* **35**, 737 (1968).
27. E. Reissner, Note on the problem of the distribution of stress in a thin stiffened elastic sheet. *Proc. of the Nat. Acad. of Sciences* **26**, 300 (1940).
28. G. C. Sih, On the Westergaard method of crack analysis. *Int. J. Fract. Mech.* **2**, 628 (1966).
29. J. Eftis and H. Liebowitz, On the modified Westergaard equations for certain plane crack problems. *Int. J. Fract. Mech.* **8**, 383 (1972).